

# THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR $s$ -CONVEX FUNCTIONS

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ABSTRACT. In this paper we introduce operator  $s$ -convex functions and establish some Hermite-Hadamard type inequalities in which some operator  $s$ -convex functions of positive operators in Hilbert spaces are involved.

Keywords: The Hermite-Hadamard inequality,  $s$ -convex functions, operator  $s$ -convex functions.

## 1. INTRODUCTION

The following inequality holds for any convex function  $f$  defined on  $\mathbb{R}$  and  $a, b \in \mathbb{R}$ , with  $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

Both inequalities hold in the reversed direction if  $f$  is concave. The inequality (1.1) is known in the literature as the Hermite-Hadamard's inequality. We note that the Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function  $f : [a, b] \rightarrow \mathbb{R}$ .

In the paper [7] Hudzik and Maligranda considered, among others, two classes of functions which are  $s$ -convex in the first and second senses. These classes are defined in the following way: a function  $f : [0, \infty) \rightarrow \mathbb{R}$  is said to be  $s$ -convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all  $x, y \in [0, \infty)$  and  $\alpha, \beta \geq 0$  with  $\alpha^s + \beta^s = 1$ . The class of  $s$ -convex functions in the first sense is usually denoted with  $K_s^1$ .

A function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  where  $\mathbb{R}^+ = [0, +\infty)$ , is said to be  $s$ -convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all  $x, y \in [0, \infty)$ ,  $\lambda \in [0, 1]$  and for some fixed  $s \in (0, 1]$ . The class of  $s$ -convex functions in the second sense is usually denoted with  $K_s^2$ . It can be easily seen that for  $s = 1$ ,  $s$ -convexity reduces to ordinary convexity of functions defined on  $[0, \infty)$ .

It is proved in [7] that if  $s \in (0, 1)$  then  $f \in K_s^2$  implies  $f([0, \infty)) \subseteq [0, \infty)$ , i.e., they proved that all functions from  $K_s^2$ ,  $s \in (0, 1)$ , are nonnegative. The following example can be found in [7].

**Example 1.** Let  $s \in (0, 1)$  and  $a, b, c \in \mathbb{R}$ . We define function  $f : [0, \infty) \rightarrow \mathbb{R}$  as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

It can be easily checked that

- (i) If  $b \geq 0$  and  $0 \leq c \leq a$ , then  $f \in K_s^2$ ,
- (ii) If  $b > 0$  and  $c < 0$ , then  $f \notin K_s^2$ .

In Theorem 4 of [7] both definitions of the  $s$ -convexity have been compared as follows:

- (i) Let  $0 < s \leq 1$ . If  $f \in K_s^2$  and  $f(0) = 0$ , then  $f \in K_s^1$ ,
- (ii) Let  $0 < s_1 < s_2 \leq 1$ . If  $f \in K_{s_2}^2$  and  $f(0) = 0$ , then  $f \in K_{s_1}^2$ ,
- (iii) Let  $0 < s_1 < s_2 \leq 1$ . If  $f \in K_{s_2}^1$  and  $f(0) \leq 0$ , then  $f \in K_{s_1}^1$ .

In [3], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which holds for  $s$ -convex functions in the second sense:

**Theorem 1.** Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is an  $s$ -convex function in the second sense, where  $s \in (0, 1)$  and let  $a, b \in [0, \infty)$ ,  $a < b$ . If  $f \in L^1[a, b]$ , then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1} \quad (1.2)$$

the constant  $k = \frac{1}{s+1}$  is the best possible in the second inequality in (1.2). The above inequalities are sharp.

The Hermite-Hadamard inequality has several applications in non-linear analysis and the geometry of Banach spaces, see [8]. In recent years several extensions and generalizations have been considered for classical convexity. We would like to refer the reader to [2, 5, 13] and references therein for more information. A number of papers have been written on this inequality providing some inequalities analogous to Hadamard's inequality given in (1.1) involving two convex functions, see [11, 1, 12]. Pachpatte in [11] has proved the following theorem for the product of two convex functions.

**Theorem 2.** *Let  $f$  and  $g$  be real-valued, nonnegative and convex functions on  $[a, b]$ . Then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b),$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b),$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

Kirmaci et al. in [9] have proved the following theorem for the product of two  $s$ -convex functions, which is a generalization of Theorem 2.

**Theorem 3.** *Let  $f, g : [0, \infty) \rightarrow [0, \infty)$  be  $s_1$ -convex and  $s_2$ -convex functions in the second sense respectively, where  $s_1, s_2 \in (0, 1)$ . Let  $a, b \in [0, \infty)$ ,  $a < b$ . If  $f, g$  and  $fg \in L^1([a, b])$  then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s_1 + s_2 + 1}M(a, b) + \beta(s_1 + 1, s_2 + 1)N(a, b).$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

In this paper we show that Theorem 1 and Theorem 3 hold for operator  $s$ -convex functions in a convex subset  $K$  of  $B(H)^+$  the set of positive operators in  $B(H)$ . We also obtain some integral inequalities for the product of two operator  $s$ -convex functions.

## 2. OPERATOR $s$ -CONVEX FUNCTIONS

First, we review the operator order in  $B(H)$  and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators  $A, B \in B(H)$  we write  $A \leq B$  (or  $B \geq A$ ) if  $\langle Ax, x \rangle \leq \langle Bx, x \rangle$  for every vector  $x \in H$ , we call it the operator order.

Now, let  $A$  be a bounded selfadjoint linear operator on a complex Hilbert space  $(H; \langle \cdot, \cdot \rangle)$  and  $C^*(Sp(A))$  the  $C^*$ -algebra of all continuous complex-valued functions on the spectrum of  $A$ . The Gelfand map establishes a  $*$ -isometrically isomorphism  $\Phi$  between  $C^*(Sp(A))$  and the  $C^*$ -algebra  $C^*(A)$  generated by  $A$  and the identity operator  $1_H$  on  $H$ .

as follows (see for instance [6, p.3]): For  $f, g \in C(Sp(A))$  and  $\alpha, \beta \in \mathbb{C}$

- (i)  $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ ;
- (ii)  $\Phi(fg) = \Phi(f)\Phi(g)$  and  $\Phi(f^*) = \Phi(f)^*$ ;
- (iii)  $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$ ;
- (iv)  $\Phi(f_0) = 1$  and  $\Phi(f_1) = A$ , where  $f_0(t) = 1$  and  $f_1(t) = t$ , for  $t \in Sp(A)$ .

If  $f$  is a continuous complex-valued functions on  $Sp(A)$ , the element  $\Phi(f)$  of  $C^*(A)$  is denoted by  $f(A)$ , and we call it the continuous functional calculus for a bounded selfadjoint operator  $A$ .

If  $A$  is a bounded selfadjoint operator and  $f$  is a real-valued continuous function on  $Sp(A)$ , then  $f(t) \geq 0$  for any  $t \in Sp(A)$  implies that  $f(A) \geq 0$ , i.e.,  $f(A)$  is a positive operator on  $H$ . Moreover, if both  $f$  and  $g$  are real-valued functions on  $Sp(A)$  such that  $f(t) \leq g(t)$  for any  $t \in sp(A)$ , then  $f(A) \leq g(A)$  in the operator order in  $B(H)$ .

A real valued continuous function  $f$  on an interval  $I$  is said to be operator convex (operator concave) if

$$f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda)f(A) + \lambda f(B)$$

in the operator order in  $B(H)$ , for all  $\lambda \in [0, 1]$  and for every bounded self-adjoint operators  $A$  and  $B$  in  $B(H)$  whose spectra are contained in  $I$ .

As examples of such functions, we give the following examples, another proof of them and further examples can be found in [6].

**Example 2.** (i) *The convex function  $f(t) = \alpha t^2 + \beta t + \gamma$  ( $\alpha \geq 0, \beta, \gamma \in \mathbb{R}$ ) is operator convex on every interval. To see it, for all self-adjoint operators  $A$  and  $B$ :*

$$\begin{aligned} \frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) &= \alpha \left( \frac{A^2 + B^2}{2} - \left(\frac{A+B}{2}\right)^2 \right) \\ &\quad + \beta \left( \frac{A+B}{2} - \frac{A+B}{2} \right) + (\gamma - \gamma) \\ &= \frac{\alpha}{4}(A^2 + B^2 - AB - BA) = \frac{\alpha}{4}(A - B)^2 \geq 0. \end{aligned}$$

(ii) *The convex function  $f(t) = t^3$  on  $[0, \infty)$  is not operator convex. In fact, if we put*

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$\frac{A^3 + B^3}{2} - \left( \frac{A + B}{2} \right)^3 = \frac{1}{8} \begin{bmatrix} 67 & -34 \\ -34 & 17 \end{bmatrix} \not\geq 0.$$

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [6] and the references therein.

We denoted by  $B(H)^+$  the set of all positive operators in  $B(H)$  and

$$C(H) := \{A \in B(H)^+ : AB + BA \geq 0, \text{ for all } B \in B(H)^+\}.$$

It is obvious that  $C(H)$  is a closed convex cone in  $B(H)$ .

**Definition 1.** Let  $I$  be an interval in  $[0, \infty)$  and  $K$  be a convex subset of  $B(H)^+$ . A continuous function  $f : I \rightarrow \mathbb{R}$  is said to be operator  $s$ -convex on  $I$  for operators in  $K$  if

$$f((1 - \lambda)A + \lambda B) \leq (1 - \lambda)^s f(A) + \lambda^s f(B)$$

in the operator order in  $B(H)$ , for all  $\lambda \in [0, 1]$  and for every positive operators  $A$  and  $B$  in  $K$  whose spectra are contained in  $I$  and for some fixed  $s \in (0, 1]$ . For  $K = B(H)^+$  we say  $f$  is operator  $s$ -convex on  $I$ .

First of all we state the following lemma.

**Lemma 1.** If  $f$  is operator  $s$ -convex on  $[0, \infty)$  for operators in  $K$ , then  $f(A)$  is positive for every  $A \in K$ .

*Proof.* For  $A \in K$ , we have

$$f(A) = f\left(\frac{A}{2} + \frac{A}{2}\right) \leq \left(\frac{1}{2}\right)^s f(A) + \left(\frac{1}{2}\right)^s f(A) = 2^{1-s} f(A).$$

This implies that  $(2^{1-s} - 1)f(A) \geq 0$  and so  $f(A) \geq 0$ .  $\square$

In [10], Moslehian and Najafi proved the following theorem for positive operators as follows:

**Theorem 4.** Let  $A, B \in B(H)^+$ . Then  $AB + BA$  is positive if and only if  $f(A + B) \leq f(A) + f(B)$  for all non-negative operator monotone functions  $f$  on  $[0, \infty)$ .

As an example of operator  $s$ -convex function, we give the following example.

**Example 3.** Since for every positive operators  $A, B \in C(H)$ ,  $AB + BA \geq 0$ , utilizing Theorem 4 we get

$$((1 - t)A + tB)^s \leq (1 - t)^s A^s + t^s B^s.$$

Therefore the continuous function  $f(t) = t^s$  ( $0 < s \leq 1$ ) is operator  $s$ -convex on  $[0, \infty)$  for operators in  $C(H)$ .

Dragomir in [4] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

**Theorem 5.** *Let  $f : I \rightarrow \mathbb{R}$  be an operator convex function on the interval  $I$ . Then for all selfadjoint operators  $A$  and  $B$  with spectra in  $I$  we have the inequality*

$$\begin{aligned} \left( f \left( \frac{A+B}{2} \right) \leq \right) & \frac{1}{2} \left[ f \left( \frac{3A+B}{4} \right) + f \left( \frac{A+3B}{4} \right) \right] \\ & \leq \int_0^1 f((1-t)A + tB) dt \\ & \leq \frac{1}{2} \left[ f \left( \frac{A+B}{2} \right) + \frac{f(A) + f(B)}{2} \right] \left( \leq \frac{f(A) + f(B)}{2} \right). \end{aligned}$$

Let  $X$  be a vector space,  $x, y \in X$ ,  $x \neq y$ . Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

We consider the function  $f : [x, y] \rightarrow \mathbb{R}$  and the associated function

$$\begin{aligned} g(x, y) : [0, 1] & \rightarrow \mathbb{R}, \\ g(x, y)(t) & := f((1-t)x + ty), t \in [0, 1]. \end{aligned}$$

Note that  $f$  is convex on  $[x, y]$  if and only if  $g(x, y)$  is convex on  $[0, 1]$ . For any convex function defined on a segment  $[x, y] \in X$ , we have the Hermite-Hadamard integral inequality

$$f \left( \frac{x+y}{2} \right) \leq \int_0^1 f((1-t)x + ty) dt \leq \frac{f(x) + f(y)}{2},$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function  $g(x, y) : [0, 1] \rightarrow \mathbb{R}$ .

**Lemma 2.** *Let  $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$  be a continuous function on the interval  $I$ . Then for every two positive operators  $A, B \in K \subseteq B(H)^+$  with spectra in  $I$  the function  $f$  is operator  $s$ -convex for operators in  $[A, B] := \{(1-t)A + tB : t \in [0, 1]\}$  if and only if the function  $\varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R}$  defined by*

$$\varphi_{x,A,B} = \langle f((1-t)A + tB)x, x \rangle$$

*is  $s$ -convex on  $[0, 1]$  for every  $x \in H$  with  $\|x\| = 1$ .*

*Proof.* Let  $f$  be operator  $s$ -convex for operators in  $[A, B]$  then for any  $t_1, t_2 \in [0, 1]$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$  we have

$$\begin{aligned}\varphi_{x,A,B}(\alpha t_1 + \beta t_2) &= \langle f((1 - (\alpha t_1 + \beta t_2))A + (\alpha t_1 + \beta t_2)B)x, x \rangle \\ &= \langle f(\alpha[(1 - t_1)A + t_1B] + \beta[(1 - t_2)A + t_2B])x, x \rangle \\ &\leq \alpha^s \langle f((1 - t_1)A + t_1B)x, x \rangle + \beta^s \langle f((1 - t_2)A + t_2B)x, x \rangle \\ &= \alpha^s \varphi_{x,A,B}(t_1) + \beta^s \varphi_{x,A,B}(t_2).\end{aligned}$$

showing that  $\varphi_{x,A,B}$  is a  $s$ -convex function on  $[0, 1]$ .

Let now  $\varphi_{x,A,B}$  be  $s$ -convex on  $[0, 1]$ , we show that  $f$  is operator  $s$ -convex for operators in  $[A, B]$ . For every  $C = (1 - t_1)A + t_1B$  and  $D = (1 - t_2)A + t_2B$  in  $[A, B]$  we have

$$\begin{aligned}\langle f(\lambda C + (1 - \lambda)D)x, x \rangle &= \langle f[\lambda((1 - t_1)A + t_1B) + (1 - \lambda)((1 - t_2)A + t_2B)]x, x \rangle \\ &= \langle f[(1 - (\lambda t_1 + (1 - \lambda)t_2))A + (\lambda t_1 + (1 - \lambda)t_2)B]x, x \rangle \\ &= \varphi_{x,A,B}(\lambda t_1 + (1 - \lambda)t_2) \\ &\leq \lambda^s \varphi_{x,A,B}(t_1) + (1 - \lambda)^s \varphi_{x,A,B}(t_2) \\ &= \lambda^s \langle f((1 - t_1)A + t_1B)x, x \rangle + (1 - \lambda)^s \langle f((1 - t_2)A + t_2B)x, x \rangle \\ &\leq \lambda^s \langle f(C)x, x \rangle + (1 - \lambda)^s \langle f(D)x, x \rangle.\end{aligned}$$

□

The following theorem is a generalization of Theorem 1 for operator  $s$ -convex functions.

**Theorem 6.** *Let  $f : I \rightarrow \mathbb{R}$  be an operator  $s$ -convex function on the interval  $I \subseteq [0, \infty)$  for operators in  $K \subseteq B(H)^+$ . Then for all positive operators  $A$  and  $B$  in  $K$  with spectra in  $I$  we have the inequality*

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)dt \leq \frac{f(A) + f(B)}{s+1}. \quad (2.1)$$

*Proof.* For  $x \in H$  with  $\|x\| = 1$  and  $t \in [0, 1]$ , we have

$$\langle ((1-t)A + tB)x, x \rangle = (1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle \in I, \quad (2.2)$$

since  $\langle Ax, x \rangle \in Sp(A) \subseteq I$  and  $\langle Bx, x \rangle \in Sp(B) \subseteq I$ .

Continuity of  $f$  and (2.2) imply that the operator-valued integral  $\int_0^1 f((1-t)A + tB)dt$  exists.

Since  $f$  is operator  $s$ -convex, therefore for  $t$  in  $[0, 1]$  and  $A, B \in K$  we have

$$f((1-t)A + tB) \leq (1-t)^s f(A) + t^s f(B). \quad (2.3)$$

Integrating both sides of (2.3) over  $[0, 1]$  we get the following inequality

$$\int_0^1 f((1-t)A + tB)dt \leq \frac{f(A) + f(B)}{s+1}.$$

To prove the first inequality in (2.1) we observe that

$$f\left(\frac{A+B}{2}\right) \leq \frac{f(tA + (1-t)B) + f((1-t)A + tB)}{2^s}. \quad (2.4)$$

Integrating the inequality (2.4) over  $t \in [0, 1]$  and taking into account that

$$\int_0^1 f(tA + (1-t)B)dt = \int_0^1 f((1-t)A + tB)dt$$

then we deduce the first part of (2.1).  $\square$

Let  $f : I \rightarrow \mathbb{R}$  be operator  $s_1$ -convex and  $g : I \rightarrow \mathbb{R}$  operator  $s_2$ -convex function on the interval  $I$ . Then for all positive operators  $A$  and  $B$  on a Hilbert space  $H$  with spectra in  $I$ , we define real functions  $M(A, B)$  and  $N(A, B)$  on  $H$  by

$$M(A, B)(x) = \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle \quad (x \in H),$$

$$N(A, B)(x) = \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle \quad (x \in H).$$

We note that, the Beta and Gamma functions are defined respectively, as follows:

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad x > 0, y > 0$$

and

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad x > 0.$$

The following theorem is a generalization of Theorem 3 for operator  $s$ -convex functions.

**Theorem 7.** *Let  $f : I \rightarrow \mathbb{R}$  be operator  $s_1$ -convex and  $g : I \rightarrow \mathbb{R}$  operator  $s_2$ -convex function on the interval  $I$  for operators in  $K \subseteq B(H)^+$ . Then for all positive operators  $A$  and  $B$  in  $K$  with spectra in  $I$ , the inequality*

$$\begin{aligned} & \int_0^1 \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle dt \\ & \leq \frac{1}{s_1 + s_2 + 1} M(A, B)(x) + \beta(s_1 + 1, s_2 + 1) N(A, B)(x). \end{aligned} \quad (2.5)$$

holds for any  $x \in H$  with  $\|x\| = 1$ .



*Proof.* For  $x \in H$  with  $\|x\| = 1$  and  $t \in [0, 1]$ , we have

$$\langle (tA + (1-t)B)x, x \rangle = t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle \in I, \quad (2.6)$$

since  $\langle Ax, x \rangle \in Sp(A) \subseteq I$  and  $\langle Bx, x \rangle \in Sp(B) \subseteq I$ .

Continuity of  $f, g$  and (2.6) imply that the operator valued integrals  $\int_0^1 f(tA + (1-t)B)dt$ ,  $\int_0^1 g(tA + (1-t)B)dt$  and  $\int_0^1 (fg)(tA + (1-t)B)dt$  exist.

Since  $f$  is operator  $s_1$ -convex and  $g$  is operator  $s_2$ -convex, therefore for  $t$  in  $[0, 1]$  and  $x \in H$  we have

$$\langle f(tA + (1-t)B)x, x \rangle \leq \langle (t^{s_1}f(A) + (1-t)^{s_1}f(B))x, x \rangle, \quad (2.7)$$

$$\langle g(tA + (1-t)B)x, x \rangle \leq \langle (t^{s_2}g(A) + (1-t)^{s_2}g(B))x, x \rangle. \quad (2.8)$$

From (2.7) and (2.8) we obtain

$$\begin{aligned} & \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\ & \leq t^{s_1+s_2} \langle f(A)x, x \rangle \langle g(A)x, x \rangle + (1-t)^{s_1+s_2} \langle f(B)x, x \rangle \langle g(B)x, x \rangle \\ & \quad + t^{s_1}(1-t)^{s_2} [\langle f(A)x, x \rangle \langle g(B)x, x \rangle] \\ & \quad + t^{s_2}(1-t)^{s_1} [\langle f(B)x, x \rangle \langle g(A)x, x \rangle]. \end{aligned} \quad (2.9)$$

Integrating both sides of (2.9) over  $[0, 1]$  we get the required inequality (2.5).  $\square$

The following theorem is a generalization of Theorem 7 in [9] for operator  $s$ -convex functions.

**Theorem 8.** *Let  $f : I \rightarrow \mathbb{R}$  be operator  $s_1$ -convex and  $g : I \rightarrow \mathbb{R}$  be  $s_2$ -convex function on the interval  $I$  for operators in  $K \subseteq B(H)^+$ . Then for all positive operators  $A$  and  $B$  in  $K$  with spectra in  $I$ , the inequality*

$$\begin{aligned} & 2^{s_1+s_2-1} \left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\ & \leq \int_0^1 \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle dt \\ & \quad + \beta(s_1 + 1, s_2 + 1)M(A, B)(x) + \frac{1}{s_1 + s_2 + 1}N(A, B)(x), \end{aligned} \quad (2.10)$$

holds for any  $x \in H$  with  $\|x\| = 1$ .

*Proof.* Since  $f$  is operator  $s_1$ -convex and  $g$  operator  $s_2$ -convex, therefore for any  $t \in I$  and any  $x \in H$  with  $\|x\| = 1$  we observe that

$$\begin{aligned}
& \left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\
&= \left\langle f\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x \right\rangle \\
&\quad \times \left\langle g\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x \right\rangle \\
&\leq \frac{1}{2^{s_1+s_2}} \left\{ \langle f(tA + (1-t)B)x, x \rangle + \langle f((1-t)A + tB)x, x \rangle \right. \\
&\quad \times [\langle g(tA + (1-t)B)x, x \rangle + \langle g((1-t)A + tB)x, x \rangle] \Big\} \\
&\leq \frac{1}{2^{s_1+s_2}} \left\{ [\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \right. \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] \\
&+ [t^{s_1} \langle f(A)x, x \rangle + (1-t)^{s_1} \langle f(B)x, x \rangle] [(1-t)^{s_2} \langle g(A)x, x \rangle + t^{s_2} \langle g(B)x, x \rangle] \\
&+ [(1-t)^{s_1} \langle f(A)x, x \rangle + t^{s_1} \langle f(B)x, x \rangle] [t^{s_2} \langle g(A)x, x \rangle + (1-t)^{s_2} \langle g(B)x, x \rangle] \Big\} \\
&= \frac{1}{2^{s_1+s_2}} \left\{ [\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \right. \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] \\
&+ (t^{s_1}(1-t)^{s_2} + t^{s_2}(1-t)^{s_1}) [\langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle] \\
&+ ((1-t)^{s_1+s_2} + t^{s_1+s_2}) [\langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle] \Big\}.
\end{aligned}$$

By integration over  $[0,1]$ , we obtain

$$\begin{aligned}
& \left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\
&\leq \frac{1}{2^{s_1+s_2}} \left( \int_0^1 [\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \right. \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] dt \\
&\quad \left. + 2\beta(s_1 + 1, s_2 + 1)M(A, B)(x) + \frac{2}{s_1 + s_2 + 1}N(A, B)(x) \right).
\end{aligned}$$

This implies the required inequality (2.10).  $\square$

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